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HEAT AND MASS TRANSFER IN POROUS AND DISPERSION MEDIA

EXTRACTION FROM A POROUS BODY IN THE PRESENCE OF PERIODIC FLUID FLOW ON IT

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This paper considers the nonstationary process of extraction from a solid body modeled by a system of semi-infinite capillaries connected with a group of no-flow channels when the mass transfer velocity in the flow is composed of two components — a constant velocity component and a time-periodic addition to the first one that is assumed to be small relative to the amplitude. We have obtained analytical dependences for the mass-transfer characteristics that are of practical interest: the concentration and the diffusion flow for both the main approximation and with correction for the periodic action on the system.

Keywords: *two-component model, mass transfer, periodicity, porous body, extraction.*

Introduction. In describing the extraction of specific components (SC) from porous structures and other mass-transfer processes in dispersion media, the diffusion model and its improved modifications have found the widest application [1–9]. As a rule, either the pure diffusion extraction (sometimes with an effective constant diffusion coefficient taking into account, integrally, the velocity field) is considered or the convective transfer is analyzed in the simplest way — assuming the transfer velocity to be constant and sometimes [10] the velocity profile to be linear. Nevertheless, recent years have seen a rising interest in various nonstationary actions on dispersion systems of both the pulsed type [11, 12] and the periodic type [13, 14].

The present work considers some cases of periodic action on the mass transfer in a porous structure of a certain type consisting of two continua interacting in terms of the mass transfer.

The two-continuum (two-phase, two-component) model of a porous material implies that at a given point in space two continua, in each of which the SC mass transfer occurs by its own specific law, are present simultaneously. Moreover, the SC from one phase can make a transition to the second one and back by a certain law.

The most rigorous approaches to the derivation of transfer equations in the two interpenetrating continua presuppose the realization of a certain procedure of averaging the transfer equations in the phases [4, 6, 7, 15]. A simpler approach is to construct a certain model of a porous medium as an element of the averaged structure and then derive the equation of the process by traditional methods on the basis of the chosen model of the porous material. Experience in modeling processes in a porous medium shows that the resulting equations are often of the same form. Here we will make use of the second approach.

Formulation of the Problem. The model of the porous material is schematically represented in Fig. 1. The formulation of problems for such a system was discussed earlier in [8–10] for both small and large times. Therefore, in investigating the initial stage of the process we shall use postulates of the cited works without additional comments. Let us assume the transfer equation in the flow channel (Fig. 1) to be one-dimensional along the spatial coordinates:

$$\left\{\frac{\partial}{\partial t} + U\,[1 + \varepsilon \sin(\Omega t)]\,\frac{\partial}{\partial Y}\right\} C\,(Y, t) = Q\,, \tag{1}$$

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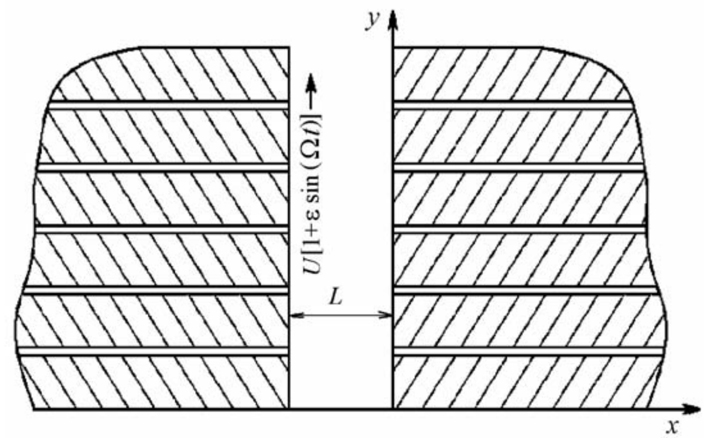

Fig. 1. Scheme of pores in a porous material.

where Q describes the CS source (ingress of SCs from stagnant capillaries). Here the second term on the left-hand side defines the convective mass transfer along the Y axis with velocity $U\,[1+\varepsilon\,\sin\,(\Omega t)]$, where ε is assumed by us to be a small quantity $|\varepsilon| \ll 1$. The diffusion transfer of SCs along the Y axis is neglected.

Under the conditions of a porous structure the periodic action on the flow in it, because of the fluid viscosity and the change in the flow direction, will be weaker in amplitude than for the unidirectional flow caused by a pressure difference of the same order as for the periodic variant. Therefore, it is natural to consider the parameter ε to be a small quantity.

Consider only the initial stage of the process while the influence of the length finiteness of stagnant capillaries is weak and this factor can be neglected. This enables us to use for the SC source Q the simplified expression (for a semi-infinite body) known in the literature (e.g., [8, 16, 17]) and reduce Eq. (1) to the form

$$\left\{\frac{\partial}{\partial T} + U\,[1+\varepsilon\,\sin\,(\Omega t)]\,\frac{\partial}{\partial Y}\right\} C\,(Y, t) = \frac{\psi}{L}\sqrt{\frac{D}{\pi}}\,\frac{\partial}{\partial T}\int_0^t \frac{[C_0 - C\,(Y, \xi)]}{\sqrt{t-\xi}}\,d\xi\,. \tag{2}$$

For the case where $\varepsilon = 0$, Eq. (2) was proposed in [18]. In fact, it represents a special case of the equation from [19] if we neglect the diffusion term, as well as assume the pores to be infinitely extended and having, at the initial instant of time, a uniform (C_0 = const) distribution of SCs along the pore. It should be noted that both the results of [18] and our investigation have a time-limited character, and the asymptotic properties of the solution at $t \to \infty$ should be interpreted as intermediate asymptotics.

In the region of the flow portion (of the channel), we assume that the SC is absent at $t = 0$. This corresponds to the initial condition for Eq. (2)

$$C\,(Y, 0) = 0\,. \tag{3}$$

To complete the formulation of the problem, it is necessary to formulate the boundary condition for Eq. (2). We assume that at the inlet to the system ($Y = 0$) the SC component is absent, i.e., we take the boundary conditions

$$C\,(0, t) = 0\,. \tag{4}$$

Equation (2) for $C = C(Y, t)$ can be considered in a bounded region $Y \in (0, Y_*)$ in an infinite time interval $t \in (0, \infty)$ (although it is necessary to take into account that the model describes the process only at the initial stage). It is expedient to write system (2)–(4) in dimensionless coordinates:

$$\tau = \frac{t}{T}\,,\quad G = \frac{C}{C_0}\,,\quad \eta = \frac{Y}{UT}\,,\quad \omega = \Omega T\,,\quad T = \frac{L^2}{\psi D}\,, \tag{5}$$

With formulas (5) taken into account, expressions (3) and (4) retain their form while Eq. (2) will be rewritten as follows:

$$\left\{\frac{\partial}{\partial\tau}+[1+\varepsilon\sin(\omega\tau)]\frac{\partial}{\partial\eta}\right\}G(\eta,\tau)=\frac{1}{\sqrt{\pi}}\frac{\partial}{\partial\tau}\int_0^\tau\frac{[1-G(\eta,\xi)]}{\sqrt{\tau-\xi}}d\xi\,. \tag{6}$$

The proposed model should be considered as a strongly simplified variant of the description of the external periodic action on the process of SC extraction in porous systems. First, the structure of the medium has been postulated as a system of capillaries (Fig. 1). Second, the fluid phase flow in a typical situation is organized by the pressure change at the boundary of the region, which, as is known, even in the simplest variant of a circular tube, leads to more complex velocity profiles [20] compared to the distribution used here and also in [13]. Third, we consider the mass transfer between the channels corresponding to the initial stage of the process. Therefore, the various asymptotics at unlimited time increase should be interpreted as intermediate [21] and having a bounded ineligibility region. In so doing, different cases of approximation inhomogeneity are possible [22]. Finally, fourthly, we make use of the method of small parameter, which by itself theoretically provides the investigation with the use of only certain corrections to the main approximation. Arguing in favor of such analysis is the fact that in systems that are difficult to describe in geometrical and physico-chemical terms it is hardly reasonable to use too complicated computing methods.

Analysis of the Problem. Let us seek a solution of problem (6), (3), (4) by the method of small perturbations [22] for the parameter ε:

$$G(\eta,\tau)=G_0(\eta,\tau)+\varepsilon G_1(\eta,\tau)+\varepsilon^2G_2(\eta,\tau)+\dots\,. \tag{7}$$

Substituting expansions (7) into Eq. (6) and using the additional conditions (3), (4), we obtain upon grouping terms of the same order in ε for the main approximation

$$\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\eta}\right)G_0(\eta,\tau)=\frac{1}{\sqrt{\pi}}\frac{\partial}{\partial\tau}\int_0^\tau\frac{[1-G_0(\eta,\xi)]}{\sqrt{\tau-\xi}}d\xi\,,\quad G_0(\eta,0)=0\,,\quad G_0(0,\tau)=0\,, \tag{8}$$

and the problem for calculating the correction to the main approximation

$$\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\eta}\right)G_1(\eta,\tau)+\sin(\omega\tau)\frac{\partial G_0(\eta,\tau)}{\partial\eta}=\frac{-1}{\sqrt{\pi}}\frac{\partial}{\partial\tau}\int_0^\tau\frac{G_1(\eta,\xi)}{\sqrt{\tau-\xi}}d\xi\,,\quad G_1(\eta,0)=0\,,\quad G_0(0,\tau)=0\,. \tag{9}$$

The solution of Eq. (8) at the above additional conditions is sought by the operational technique. Let us apply to Eq. (8) the Laplace transform [23, 24] of the variable τ:

$$f^*(p)=\int_0^\infty\exp(-p\tau)f(\tau)\,d\tau\,.$$

As a result, we get

$$\left(\frac{d}{d\eta}+p+\sqrt{p}\right)G_0^*=\frac{1}{\sqrt{p}}\,, \tag{10}$$

$$G_0^*=G_0^*(\eta,p)\,,\quad \eta\in(0,\infty)\,,\quad G_0^*(0,p)=0\,. \tag{11}$$

The solution of the ordinary differential equation (10) at condition (11) has the form

$$G_0^*(\eta,p)=\frac{1-\exp[-(p+\sqrt{p})\,\eta]}{\sqrt{p}\,(p+\sqrt{p})}\,. \tag{12}$$

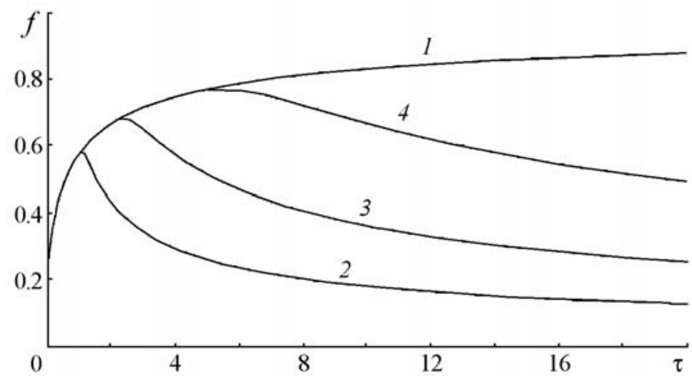

Fig. 2. Kinetic curves for the chosen space point $f = G_0(\eta, \tau)$: 1) upper line of formula (13); 2) $\eta = 1$; 3) 2; 4) 4.

With some standard computations, expression (12) can be transformed as

$$G_0(\eta, \tau) = 1 - \exp(\tau)\,\mathrm{erfc}\,(\sqrt{\tau}\,), \quad \tau < \eta,$$
$$G_0(\eta, \tau) = \exp(\tau)\left[\mathrm{erf}\,(\sqrt{\tau}\,) - \mathrm{erf}\left(\frac{2\tau - \eta}{2\sqrt{\tau - \eta}}\right)\right] + \mathrm{erf}\left(\frac{\eta}{2\sqrt{\tau - \eta}}\right), \quad \tau > \eta, \tag{13}$$

where $\mathrm{erf} = 1 - \mathrm{erfc}\,(z)$ is the probability integral $\left(\mathrm{erf}\,(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-\xi^2)\, d\xi\right)$ for which there are detailed tables and convenient computational formulas. Moreover, this function is represented in standard programs (e.g., in MathCad).

Solution (13) has a wave character. However, this can easily be determined also from formula (12) with the use of the properties of correspondence between originals and images of functions in the operational calculus [23, 24]. The wave character of the solution of the given problem was not noticed in [18]. Solution (13) changes its form at some point Y after a disturbance (influence of the boundary conditions) propagating at velocity U (unit velocity in dimensionless coordinates in the main approximation for ε) arrives there. At $\tau < \eta$ the solution is exactly the same as in a usual "thermal" vessel. This means that in the given region the solution is determined by the initial condition alone, which can be seen from the pattern of the field of characteristics in the principal part of Eq. (8) as well as from solution (13) (first line). When $\tau > \eta$ a wave arrives at the chosen point and the solution changes its form (second line of (13)). Note that if we consider Eq. (2) in the range $-\infty < Y < \infty$ under the same condition for time at $t = 0$, then the solution of the problem in the main approximation for ε will be given by the third line of (13) (at $\tau < \eta$), i.e., in this case the solution does not depend on the Y coordinate.

Figures 2 and 3 present the plots of functions characterizing the change with time in the SC concentration determined in the main approximation of (13) at a given space point (Fig. 2) and the SC concentration profile at chosen instants of time (Fig. 3). Curve 1 in Fig. 2 is defined by the upper line of formula (13) and its portions are common for all kinetic curves. Noteworthily, upon arrival of the wave (transition to the lower line of (13)) the SC concentration at the chosen point continues to grow for some time. Upon reaching its maximum the SC concentration begins to decrease. The curves in Fig. 3 are characterized by the fact that before arrival of the wave ($\eta = \tau$) the SC concentration varies synchronously (at $\tau < \eta$) with time, then (at $\tau > \eta$) the synchronism is broken. This is reflected by the horizontal straight lines in Fig. 3.

Of certain interest are expansions of solution (13) at small and large values of time. The first expansion at $\tau \to 0$ represented by the upper line of relations (13) is a convergent series:

$$G_0(\eta, \tau) = \sqrt{\tau}\sum_{n=0}^{\infty}\frac{(-1)^n\,\tau^{n/2}}{\Gamma\left(\frac{3}{2} + \frac{n}{2}\right)}, \quad \tau < \eta.$$

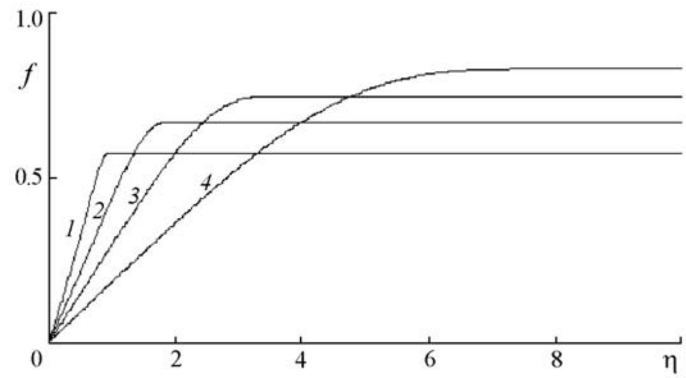

Fig. 3. SC concentration distribution in a porous body at a fixed instant of time $f = G_0(\eta, \tau)$: 1) $\tau = 1$; 2) 2; 3) 4; 4) 10.

The asymptotic, at $\tau \to \infty$, expansion of solution (13) is defined by the lower line of (13) and can be given in the following form:

$$G_0(\eta, \tau) \cong \frac{\eta}{\pi\sqrt{\tau}} \sum_{n=0}^{\infty} \Gamma\left(n + \frac{1}{2}\right)\left(\frac{\eta}{\tau}\right)^n P_n(\eta), \quad \tau > \eta, \quad \tau \to \infty, \quad \eta = O(1), \tag{14}$$

where

$$P_n(\eta) = \sum_{j=0}^{n} \frac{(-\eta)^j}{(n-j)!\,(n+j+1)\,(2j)!}, \quad P_0(\eta) = 1 .$$

In deriving the latter relations, we used the formula for the generating function of the known Hermitian polynomials [25]. By virtue of the foregoing polynomials $P_n(\eta)$ are related to the Hermitian polynomials.

Taking the Laplace transform of Eq. (9), we arrive at the relation

$$\left(\frac{\partial}{\partial\eta} + p + \sqrt{p}\right) G_1^* + \frac{1}{2i}\frac{\partial}{\partial\eta}\left[G_0^*(\eta, p - i\omega) - G_0^*(\eta, p + i\omega)\right] = 0, \tag{15}$$

where we made use of the expression for sin $(\omega\tau)$ in terms of exponents. Further, taking into account relation (12) and the initial condition (9) with account for the initial condition, we obtain after solving Eq. (15):

$$G_1^*(\eta, p) = \frac{\exp(-\eta p)}{2i}\left\{\frac{\exp\left[\eta\,(i\omega - \sqrt{p - i\omega}\,)\right] - \exp\left[-\eta\sqrt{p}\right]}{(\sqrt{p} + i\omega - \sqrt{p - i\omega}\,\sqrt{p - i\omega})} - \frac{\exp\left[-\eta\,(i\omega + \sqrt{p + i\omega}\,)\right] - \exp\left[-\eta\sqrt{p}\right]}{(\sqrt{p} - i\omega - \sqrt{p + i\omega})\,\sqrt{p + i\omega}}\right\}. \tag{16}$$

Note that because of the oddness of the correction (function sin $(\omega\tau)$ as to the parameter ω, an analogous and useful (for calculations) property of relation (16) follows: oddness as to the variable ω, which makes it possible to restrict oneself, in analysis, to one of the two terms in curly brackets in expression (16). Further we use different approximations of expression (16) and heuristic techniques to obtain approximate forms of the problem solution.

The factor exp $(-\eta p)$ in expression (16), by virtue of the operational-calculus-translation theorem [23, 24], guarantees equality to zero of the correction at $\tau < \eta$, which can also be seen from relations (9) and (13) (upper line). Likewise, it may be noted that the higher terms of expansion (7) will also be zero at $\tau < \eta$.

The above conclusion has a more general significance. The structure of Eq. (6) and condition (3) are such that a solution depending only on τ is admitted and it has the form of the upper line of (13). However, such a solution cannot be realized everywhere in the region of $\tau > 0$, $\eta > 0$ since it does not satisfy condition (4). The differential

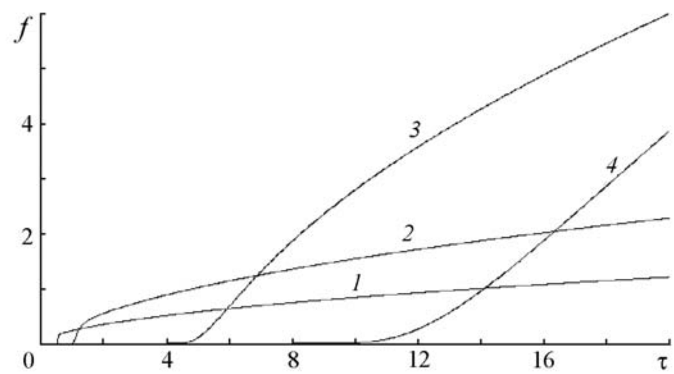

Fig. 4. Low-frequency correction to the main approximation (13) as a function of τ at various values of η 1) η = 0.5; 2) 1; 3) 4; 4) 8; $f = G_1(\eta, \tau)/\omega$.

operator on the left-hand side of (6) has a certain family of characteristics. Of particular importance is the characteristic starting from the origin of coordinates

$$\eta - \tau + \varepsilon \cos(\omega\tau)/\omega = \varepsilon/\omega . \tag{17}$$

Solution (13) (upper line) takes place when the following inequality between variables η and τ in the first quadrant of the plane η, τ holds:

$$\eta > \tau + \varepsilon\,[1 - \cos(\omega\tau)]/\omega .$$

It is natural that at the opposite sign of the inequality a solution is not found so easily (the variant $G = G(\eta)$ is excluded). Our expansion (7) in all approximations has a dividing line between the influence of the initial and boundary conditions of the form $\eta = \tau$. For the correction of the first kind under consideration, the written relations are quite sufficient.

Low-frequency approximation. Consider the case of low disturbance frequencies; more precisely, we assume that the inequality $\omega << |p|$ holds.

R e m a r k**.** The inequality $\omega << |p|$ should be understood as a marked excess of $|p|$ values in the domain of variability of the complex variable p defining the function $G_1(\eta, \tau)$ (in particular, by the Raman–Mellin inversion formula) over the parameter ω. This inequality can be compared with the heuristic rule supported by some theorems [23, 24], according to which the behavior of the original at small values of τ is determined by the behavior of the Laplace transform at large $|p|$ values and, vice versa, small values of $|p|$ influence the behavior of the original at $\tau \to \infty$. In fact, expression (16) is simplified at small values of the parameter ω, which, as is known, may lead to an inhomogeneity of expansion [22]. A similar remark also concerns the high-frequency approximation considered below.

Performing necessary calculations, from [15] we get

$$G_1^*(\eta, p) \cong -\,\omega\,\frac{\exp\left[-\,(p + \sqrt{p})\,\eta\right]}{2\sqrt{p}}\left[\frac{\eta}{p} + \eta^2\left(1 + \frac{1}{2\sqrt{p}}\right)\right] \tag{18}$$

— the first term of expansion of the function $G_1^*(\eta, p)$ by the series expansion parameter ω. The transform of expression (18) leads to the formula

$$G_1(\eta, \tau) = -\,\frac{\omega\eta^2}{2}\left[\frac{\exp(-\lambda)}{\sqrt{\pi(\tau - \eta)}} + \frac{\exp(-\lambda)}{\sqrt{\pi\lambda}} - \frac{1}{2}\operatorname{erfc}(\sqrt{\lambda})\right] H(\tau - \eta)\,, \tag{19}$$

where $H(z) = \begin{cases} 1, & z > 0, \\ 0, & z < 0. \end{cases}$ Note that the result (18), (19) (linearity of parameter ω correction) is due to the choice of the periodic addition to the velocity profile in the form of a sinusoid sin ($\omega\tau$). Otherwise the correction form will

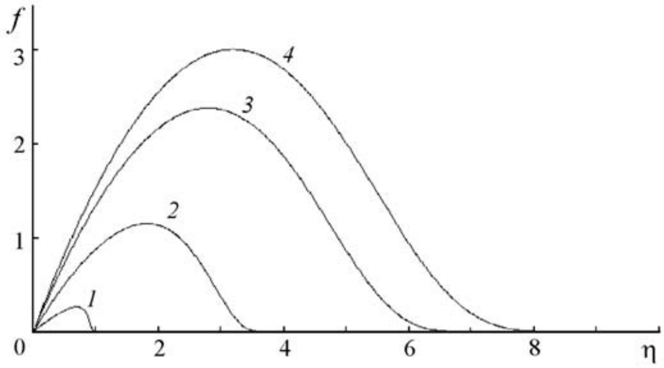

Fig. 5. Low-frequency correction to the main approximation (13) as a function of η at various values of τ: 1) τ = 1; 2) 4; 3) 8; 4) 10; $f = G_1(\eta, \tau)/\omega$.

change. In fact, the correction periodicity in the approximation of (19) does not yet manifest itself. Actually, the given correction can also be called the approximation of small values of time. This is confirmed by Fig. 4 from which it is seen that the solution disturbance does not appear periodic.

Figures 4 and 5 present the plots of functions 19, characterizing the change with time in the correction for the SC concentration at a given space point (Fig. 4) and the correction for the SC concentration profile at chosen instants of time (Fig. 5) to the main approximation (13).

High-frequency approximation. We now turn to the variant where $\omega >> |p|$ in expressions (15), (16). As above, let us find only the principal term of the given approximation. More precisely, let us transform expression (16) by using the inequality $\omega >> |p|$ where this will not lead to a too simplified result. Having performed the necessary computations, we get

$$G_1^*(\eta, p) = \frac{\exp(-\eta p)}{2i\sqrt{\omega}} \left\{ \frac{\exp\left(-\eta\gamma - \frac{i\pi}{4}\right) - \exp\left(-\frac{i\pi}{4}\right)}{\sqrt{p} - \gamma} - \frac{\exp\left(-\eta\bar{\gamma} + \frac{i\pi}{4}\right) - \exp\left(\frac{i\pi}{4}\right)}{\sqrt{p} - \bar{\gamma}} \right\}, \tag{20}$$

where the bar denotes complex conjugation. In all expressions of the present paper, by the square root of a certain quantity is meant the principal value of the root (branch), i.e., in particular, $\sqrt{1} = 1$. The transform of expression (20) leads to the expediency of introducing the following function (original corresponding to the image $1/(p^{1/2} - \gamma)$):

$$R(\gamma, \tau) = \gamma \exp(\gamma^2 \tau) \operatorname{erfc}(-\gamma\sqrt{\tau}) + \frac{1}{\sqrt{\pi\tau}},$$

in terms of which we express the function $G(\eta, \tau)$ as

$$G_1(\eta, \tau) = \frac{H(\tau - \eta)}{2i\sqrt{\omega}} \left\{ \left[\exp\left(-\eta\gamma - \frac{i\pi}{4}\right) - \exp\left(-\frac{i\pi}{4}\right) \right] R(\gamma, \tau - \eta) - \left[\exp\left(-\eta\bar{\gamma} + \frac{i\pi}{4}\right) - \exp\left(\frac{i\pi}{4}\right) \right] R(\bar{\gamma}, \tau - \eta) \right\}. \tag{21}$$

Figures 6 and 7 show the plots of functions (21) characterizing the change with time in the correction to the SC concentration at a given space point (Fig. 6) and the correction to the SC concentration profile at chosen instants of time (Fig. 7) for the main approximation (13). We chose for illustration a none too large value of $\omega = 2.4$, since at large ω values the function G_1 is noticeably simplified, $G_1(\eta, \tau) = H(\tau - \eta)[\cos(\eta\omega + \pi/4) - \cos(\pi/4)]/\omega^{3/2}$, and its plots become nonexpressive.

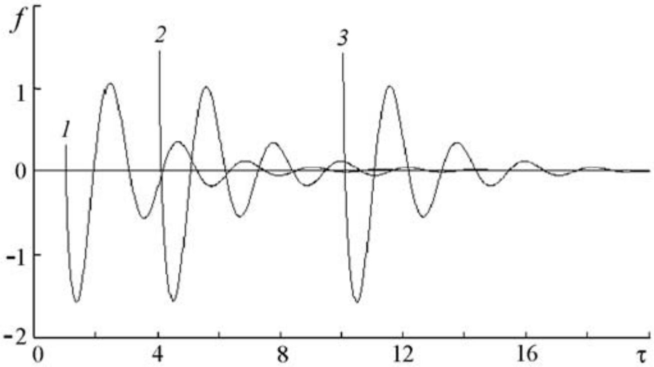

Fig. 6. High-frequency correction to the main approximation (13) as a function of τ at various values of η: 1) $\eta = 1$; 2) 4; 3) 10; $f = G_1(\eta, \tau)$.

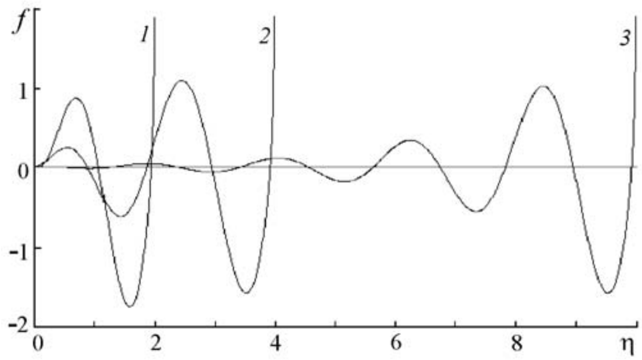

Fig. 7. High-frequency correction to the main approximation (13) as a function of η at various values of τ: 1) $\tau = 2$; 2) 4; 3) 10; $f = G_1(\eta, \tau)$.

General case. Below we will obtain a formula for solving the problem following from the inversion of relation (16) and not connected with any simplifications. In transforming expression (16) with the aid of the Raman–Mellin formula [23, 24], one should pay attention to the presence in function (16) of three branch points $p = 0$, $p = \pm i\omega$ $(\omega > 0)$ and to its asymptotic behavior at $|p| \to \infty$ in the left half-plane $\mathrm{Im}\, p < 0$ of the complex variable p (in order to make use of the Jordan lemma [23, 24]). Other singularities (poles) in function (16) are absent. For the first term in curly brackets of expression (16), the singular points will be $p = 0$ and $p = i\omega$, which can be connected by a rectilinear cut to single out a unique branch for this term. The properties of the above function are such that the path of integration in the Raman–Mellin formula can be replaced by integration with respect to the edges of the above-mentioned cut. Likewise, for the second term in relation (16), integration in the Raman–Mellin formula can be reduced to integration with respect to the edges of the rectilinear cut between the branch points $p = 0$ and $p = -i\omega$. As a result, with the above transformations and certain calculations, we arrive at the following expression for the function $G_1(\eta, \tau)$:

$$G_1(\eta, \tau) = \frac{H(\tau - \eta)}{\pi} \int_0^1 dy\, \mathrm{Im} \left\{ \exp\left[i\omega (\tau - \eta)(1 - y^2) \right] \left[\frac{\exp\left[i\eta (\omega + y\sqrt{i\omega}) \right] - \exp\left[\eta \sqrt{i\omega (1 - y^2)} \right]}{\sqrt{1 - y^2} - \sqrt{i\omega} - iy} \right.\right.$$

$$\left.\left. - \frac{\exp\left[i\eta (\omega - y\sqrt{i\omega}) \right] - \exp\left[-\eta \sqrt{i\omega (1 - y^2)} \right]}{\sqrt{1 - y^2} + \sqrt{i\omega} - iy} \right] \right\}. \qquad (22)$$

A solution of the problem in the form (22) is easy to realize by the numerical integration methods since the integrand in (22) is regular at typical values of the parameters. However, the parameters themselves can vary over a wide range. In so doing, the integrand in certain domains of variability oscillates fast and the standard numerical algorithms give no more exact results. Here one has to revert to the methods of [26]. The situation is complicated by the presence of

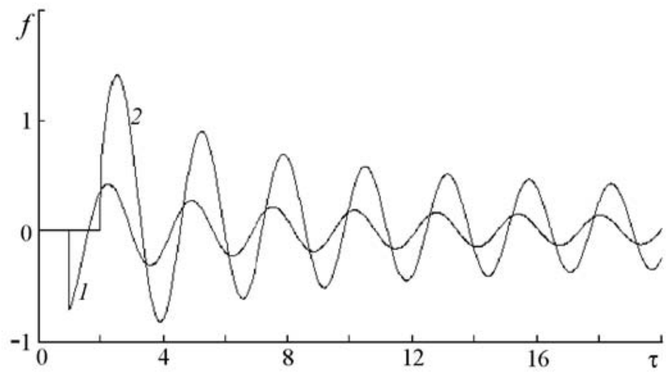

Fig. 8. Correction (22) to the main approximation (13) as a function of τ at various values of η: 1) η = 1.0; 2) 2; $f = G_1(\eta,\ \tau)$.

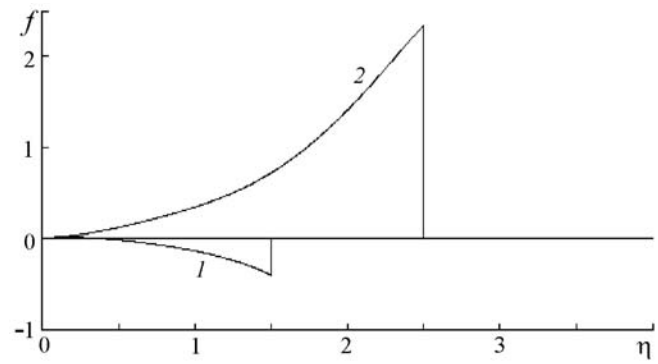

Fig. 9. Correction (22) to the main approximation (13) as a function of η at various values of τ: 1) τ = 1.5; 2) 2; $f = G_1(\eta,\ \tau)$.

the three parameters τ, η, and ω entering into the integrand. Some cases where the values of the above parameters are of the order of one and numerical solutions (in MathCad) are based on the integral of (22) at ω = 2.4 are illustrated in Figs. 8 and 9.

In the case of large values of τ and η, as well as of $\tau - \eta$ (more precisely, when the Laplace (stationary phase) method is applicable to the integral of (22)), we can obtain certain asymptotic expressions for the function $G_1(\eta,\ \tau)$. If we are primarily interested in large values of time at $\eta = O(1)$, then the value of the integral of (22) is determined by the behavior of the integrand in the vicinity of the point y = 0. The asymptotic value of relation (22) therewith is as follows:

$$G_1\,(\eta,\,\tau) = \frac{H\,(\tau-\eta)}{\sqrt{\pi}\,(\tau-\eta)}\,\mathrm{Im}\left\{\frac{\exp\,(i\omega\tau)}{1-i\omega}\left[1-\exp\,(-\,i\omega\eta)\left(\cosh\,(\eta\sqrt{i\omega}\,)+\frac{\sinh\,(\eta\sqrt{i\omega}\,)}{\sqrt{i\omega}}\right)\right]\right\}. \tag{23}$$

If the η value is also large, then it is necessary to take into account the behavior of individual terms in the integrand of (22) in the vicinity of the point of y = 1. In such an event, acting according to the standard scheme [23, 26] of the Laplace (stationary phase) method, we get

$$G_1\,(\eta,\,\tau) = \frac{H\,(\tau-\eta)}{\pi}\,\mathrm{Im}\left\{\frac{\sqrt{i}}{\eta\sqrt{\omega}}\left[\frac{\exp\,(i\omega\tau)}{1-\sqrt{i\omega}}+\frac{\exp\,(i\omega\eta)}{i-\sqrt{i\omega}}\right]-\frac{\exp\,[i\omega\,(\tau-\eta)\,\sqrt{\pi}}{(1-i\omega)\,\sqrt{\tau-\eta}}\left[\cosh\,(\eta\sqrt{i\omega}\,)+\frac{\sinh\,(\eta\sqrt{i\omega}\,)}{\sqrt{i\omega}}\right]\right\}.$$

Combining different asymptotics, we can arrive at the following formula:

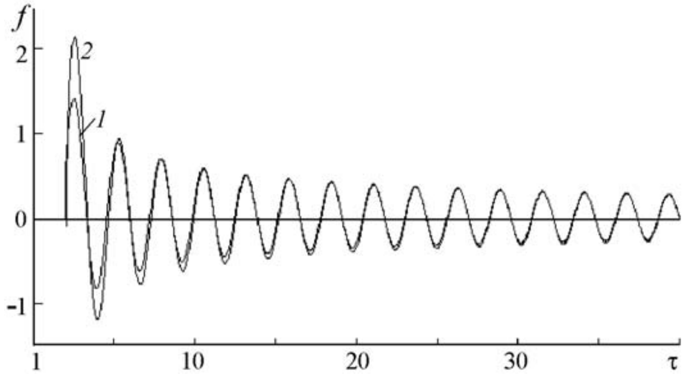

Fig. 10. Correction (22) to the main approximation (13) as a function of τ at η = 2.0; ω = 2.4: 1) exact solution of (22); 2) asymptotic solution of (23); $f = G_1(\eta, \tau)$.

$$G_1(\eta, \tau) = \frac{H(\tau-\eta)}{2\sqrt{\pi}} \operatorname{Im}\left\{\frac{2\exp(i\omega\tau)}{(1-i\omega)\sqrt{\tau-\eta}} + \frac{\exp[i\omega(\tau-\eta)]}{\sqrt{i\omega}}\left[\frac{\exp\left[-\eta\sqrt{i\omega}\right]}{(1+i\omega)\sqrt{\tau-\eta-\dfrac{\eta}{2\sqrt{i\omega}}}} - \frac{\exp\left[\eta\sqrt{i\omega}\right]}{(1-\sqrt{i\omega})\sqrt{\tau-\eta+\dfrac{\eta}{2\sqrt{i\omega}}}}\right]\right\}.$$

In a real problem, the value of the variable η is limited; therefore, in Fig. 10 only the data for formula (23) are presented. Note, however, that the last two dependences in their domains of variability of τ also agree well with the numerical value of the integral of (22). From formula (23) it is seen that the correction is a time sinusoid with an oscillation amplitude decreasing as $G_1 \cong \tau^{-1/2}$, as is the main approximation (see (14)). In such problems, the solution often becomes ineligible when $\tau \to \infty$ [22]. In fact, since the function G has the meaning of the concentration, i.e., it is positive, it is essential that the obtained binomial expansion $G(\eta, \tau) \cong G_0(\eta, \tau) + \varepsilon G_1(\eta, \tau)$ be also positive.

Figure 10 shows the curves describing the exact solution (22) (curve 1) and the asymptotic solution (23) (curve 2) at ω = 2.4. Even at a small value of η = 2 it is seen that there is a qualitative agreement between the two solutions, and at $\tau > 20$ there is a quantitative agreement (at large values of η and τ, numerical integration with the standard procedure of the MathCad system is unreliable). Note that in the ineligibility domain of relation (23) when $\tau \to \eta$, it has a power singularity.

Let us also present some formulas characterizing the SC extraction process. The SC flow density $Q(\eta, \tau)$ at the interface is described by the term on the right side of the input equation (6), as well as by the corresponding terms of the equations of the main approximation (8) and the first correction to it (9). The Laplace-transformed expression for the SC flow density determined in the main approximation has the form

$$Q_0^*(\eta, p) = \frac{\sqrt{p} + \exp\left[-(p+\sqrt{p})\,\eta\right]}{\left(p+\sqrt{p}\right)},$$

from which the functions

$$Q_0(\eta, \tau) = \frac{1}{\sqrt{\pi\tau}} - \exp(\tau)\,\mathrm{erfc}\left(\sqrt{\tau}\right), \quad \tau < \eta;$$

$$Q_0(\eta, \tau) = \frac{1}{\sqrt{\pi\tau}} + \exp(\tau)\left[\mathrm{erf}\left(\sqrt{\tau}\right) - \mathrm{erf}\left(\frac{2\tau-\eta}{2\sqrt{\tau-\eta}}\right)\right], \quad \tau > \eta \tag{24}$$

follow.

The first terms of the expansions of relation (24) at small and large times are of the form

$$Q_0(\eta,\tau)=\frac{1}{\sqrt{\pi\tau}}=-1+\ldots,\quad \tau<\eta,\quad \tau\to 0;\quad Q_0(\eta,\tau)\cong\frac{1}{\sqrt{\pi\tau}}\left(1+\frac{\eta^2}{4\tau}\right)+\ldots,\quad \tau>\eta,\quad \tau\to 0,\quad \eta=O(1).$$

For the time-integrated value of the SC flow density (density of the quantity of SCs flowing out of stagnant channels per unit area in time t), we have $M_0^*(\eta, p) = Q_0^*(\eta, p)/p$ and, accordingly, in the space of originals

$$M_0(\eta,\tau)=G_0(\eta,\tau)+\eta H(\tau-\eta)\left[\frac{1}{\sqrt{\pi\lambda}}\exp(-\lambda)-\operatorname{erfc}(\lambda)\right]. \tag{25}$$

For correction terms, we find from Eq. (9)

$$Q_1^*(\eta,p)=\frac{G_1^*(\eta,p)}{\sqrt{p}},\quad M_1^*(\eta,p)=\frac{Q_1^*(\eta,p)}{p}.$$

Using formula (18) of the low-frequency approximation and inverting the Laplace transform, we obtain

$$Q_1(\eta,\tau)=H(\tau-\eta)\left\{\frac{\eta}{2}\operatorname{erfc}\left(\frac{\eta}{2\sqrt{\tau-\eta}}\right)+\frac{\tau\eta^2}{4\sqrt{\pi}\,(\tau-\eta)^{3/2}}\exp\left(-\frac{\eta^2}{4(\tau-\eta)}\right)\right\},$$

$$M_1(\eta,\tau)=-\frac{\omega\eta H(\tau-\eta)}{4}\left[\operatorname{erfc}(\sqrt{\lambda})\,(2\tau+\eta^2)+\frac{\eta\exp(-\lambda)}{\sqrt{\pi}}\left(\frac{\eta}{\sqrt{\lambda}}-2\sqrt{\tau-\eta}\right)\right].$$

Likewise, for the high-frequency approximation

$$\begin{aligned}Q_1(\eta,\tau)&=\frac{H(\tau-\eta)}{\sqrt{\omega}}\operatorname{Im}\left\{\left[\exp\left(-\eta\gamma-\frac{i\pi}{4}\right)-\exp\left(-\frac{i\pi}{4}\right)\right]\gamma R(\gamma,\tau-\eta)\right\},\\ M_1(\eta,\tau)&=\frac{H(\tau-\eta)}{\sqrt{\omega}}\operatorname{Im}\left\{\left[\exp\left(-\eta\gamma-\frac{i\pi}{4}\right)-\exp\left(-\frac{i\pi}{4}\right)\right]R_1(\gamma,\tau-\eta)\right\},\end{aligned} \tag{26}$$

where

$$R_1(\gamma,\tau)=\exp(\gamma^2\tau)\operatorname{erfc}(-\gamma\sqrt{\tau}).$$

In deriving formula (26) from relation (20), we rejected the terms leading to the δ-function of τ which are immaterial in the considered approximation. Therefore, formulas (21) and (26) of the high-frequency approximation turned out to be similar in appearance.

By analogy with the derivation of formula (22), we obtain the following relation for the function $Q_1(\eta, \tau)$ without simplifications connected with the quantity ω:

$$Q_1(\eta,\tau)=\frac{H(\tau-\eta)\sqrt{\omega}}{\pi}\int_0^1 dy\sqrt{1-y^2}\,\operatorname{Im}\left\{\exp\left[i\omega(\tau-\eta)(1-y^2)\right]\left[\frac{\exp\left[i\eta\left(\omega+y\sqrt{i\omega}\right)\right]-\exp\left[\eta\sqrt{i\omega(1-y^2)}\right]}{\sqrt{\omega}-\sqrt{i(y^2-1)}+y\sqrt{i}}\right.\right.$$

$$\left.\left.-\frac{\exp\left[i\eta\left(\omega-y\sqrt{i\omega}\right)\right]-\exp\left[-\eta\sqrt{i\omega(1-y^2)}\right]}{\sqrt{\omega}+\sqrt{i(y^2-1)}-y\sqrt{i}}\right]\right\}. \tag{27}$$

We also present the expression for the function $M_1(\eta, \tau)$ by analogy with formulas (22), (27)

$$M_1(\eta,\tau)=\frac{H(\tau-\eta)}{\pi\sqrt{\omega}}\int\limits_0^1\frac{dy}{\sqrt{1-y^2}}\,\mathrm{Im}\left\{\exp\left[i\omega(\tau-\eta)(1-y^2)\right]\left[\frac{\exp\left[i\eta\left(\omega-y\sqrt{i\omega}\right)\right]-\exp\left[-\eta\sqrt{i\omega(1-y^2)}\right]}{\sqrt{\omega}+\sqrt{i(y^2-1)}-y\sqrt{i}}\right.\right.$$

$$\left.\left.-\frac{\exp\left[i\eta\left(\omega+y\sqrt{i\omega}\right)\right]-\exp\left[\eta\sqrt{i\omega(1-y^2)}\right]}{\sqrt{\omega}-\sqrt{i(y^2-1)}+y\sqrt{i}}\right]\right\}. \tag{28}$$

Actually, the integrals in relations (27) and (28) differ only in the provision of the function $(1-y^2)^{1/2}$: in (27) it is in the numerator and in (28) it is in the denominator, which facilitates their joint investigation. From relations (27) and (28) we obtain by the Laplace (stationary phase) method the asymptotic formulas

$$Q_1(\eta,\tau)=\frac{H(\tau-\eta)}{\sqrt{\pi(\tau-\eta)}}\,\mathrm{Im}\left\{\frac{\exp[i\omega(\tau-\eta)]}{1-i\omega}\left[\cosh\left(\eta\sqrt{i\omega}\right)+\sqrt{i\omega}\sinh\left(\eta\sqrt{i\omega}\right)-\exp(i\omega\eta)\right]\right\},$$

$$M_1(\eta,\tau)=\frac{H(\tau-\eta)}{\omega\sqrt{\pi(\tau-\eta)}}\,\mathrm{Re}\left\{\frac{\exp[i\omega(\tau-\eta)]}{i\omega-1}\left[\cosh\left(\eta\sqrt{i\omega}\right)+\sqrt{i\omega}\sinh\left(\eta\sqrt{i\omega}\right)-\exp(i\omega\eta)\right]\right\},$$

complementing relation (23). From the above formulas it is seen that the functions $Q_1(\eta,\tau)$ and $M_1(\eta,\tau)$ represent, to an accuracy of the factors, the imaginary and real parts of the expression in curly brackets.

Concluding Remarks. While the solutions of the main approximation of (13), (24), and (25) are continuous on the "wave front" $\eta = \tau$, the relations of order ε (22), (27), and (28) have a discontinuity on this line, which can be checked analytically and which is seen from the corresponding plots. Since the initial (3) and boundary (4) conditions are matched and the right side of Eq. (6) is linear with respect to G (this is a fractional half-order derivative of the function G) and contains no singular functions (of the type of the δ-function), it may be expected that the solution of the complete problem will be continuous everywhere. The discontinuity in the approximation of order ε is due to the change of the same-order equations of characteristic lines (e.g., (17)). For a more exact description of the process in the vicinity of the separating characteristic (17), a local equation can be constructed. In practice, however, of interest is the behavior of solutions at fairly large times (at an appreciable distance from the line $\eta = \tau$). Therefore, here we will not carry out a local analysis of the solution at $\eta \approx \tau$.

CONCLUSIONS

1. The proposed relations for determining the whole quantity of the specific component extracted from a porous body, the diffusion flow, and some of the other characteristics of the process can be used to describe the initial stage of extraction in the presence of a fluid flow pulsating according to the law $U[1+\varepsilon\sin(\Omega t)]$ in the flow channels of the porous medium.

2. In the main approximation for ε the solution of the problem has a wave character. At small times (with account for concrete additional conditions), all characteristics of the process depend only on time. Upon arrival of the wave the solution begins to depend also on the space coordinate.

3. The profiles of the process characteristics at a fixed instant of time at space points at which the wave has not yet arrived retain their constant values.

4. The asymptotic solutions at large times describe fairly well the first approximation for the series expansion parameter ε.

NOTATION

C, concentration, kg/m^3; C_0, SC concentration in the branching capillaries at the initial instant of time $t = 0$, kg/m^3; D, diffusion coefficient, m^2/sec; G, dimensionless SC concentration at the boundary between small and large pores; G_0, G_1, G_2, functions in decomposition (7) of dimensionless concentrations; $H(z)$, Heaviside function; Im, Re, imaginary and real part of the complex number, respectively; i, imaginary unit; L, channel width, m; M, dimensionless "yield" of the specific product through the surface by the instant of time t; M_0, M_1, main approximation and correction in the expansion of the function M; $P_n(\eta)$, polynomials of order n; p, parameter of the Laplace transform for the variable τ; Q, dimensionless instantaneous diffusion flow; Q_0, Q_1, main approximation and correction in the expansion of the function Q; R, R_1, auxiliary functions; T, natural time scale, sec; t, time, sec; U, scale of the fluid flow velocity, m/sec; Y, coordinate in the direction of the fluid flow, m; Y_*, boundary value of the Y coordinate, m; y, integration variables; $\Gamma(z)$, gamma function; $\gamma = (1 - i)(\omega/2)^{1/2} + i\omega$; ξ, integration variable; ε, series expansion parameter; η, dimensionless coordinate in the direction of the fluid flow; $\lambda = \eta^2/[4(\tau - \eta)]$; τ, dimensionless time; ψ, portion of the interface surface occupied by pore holes; Ω, ω, dimensional (sec^{-1}) and dimensionless frequency of velocity pulsations, respectively; asterisk, Laplace transform of a quantity.

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